Numerical Solution of Equations

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Numerical methods

- **Direct methods →** attempt to solve a numerical problem by a finite sequence of operations. In absence of round off errors deliver an exact solution; e.g., solving a linear system *Ax = b by Gaussian* elimination.
- **Iterative methods →** attempt to solve a numerical problem (for example, finding the root of an equation or system of equations) by finding successive approximations to the solution starting from an initial guess.
  - The stopping criteria: the relative error

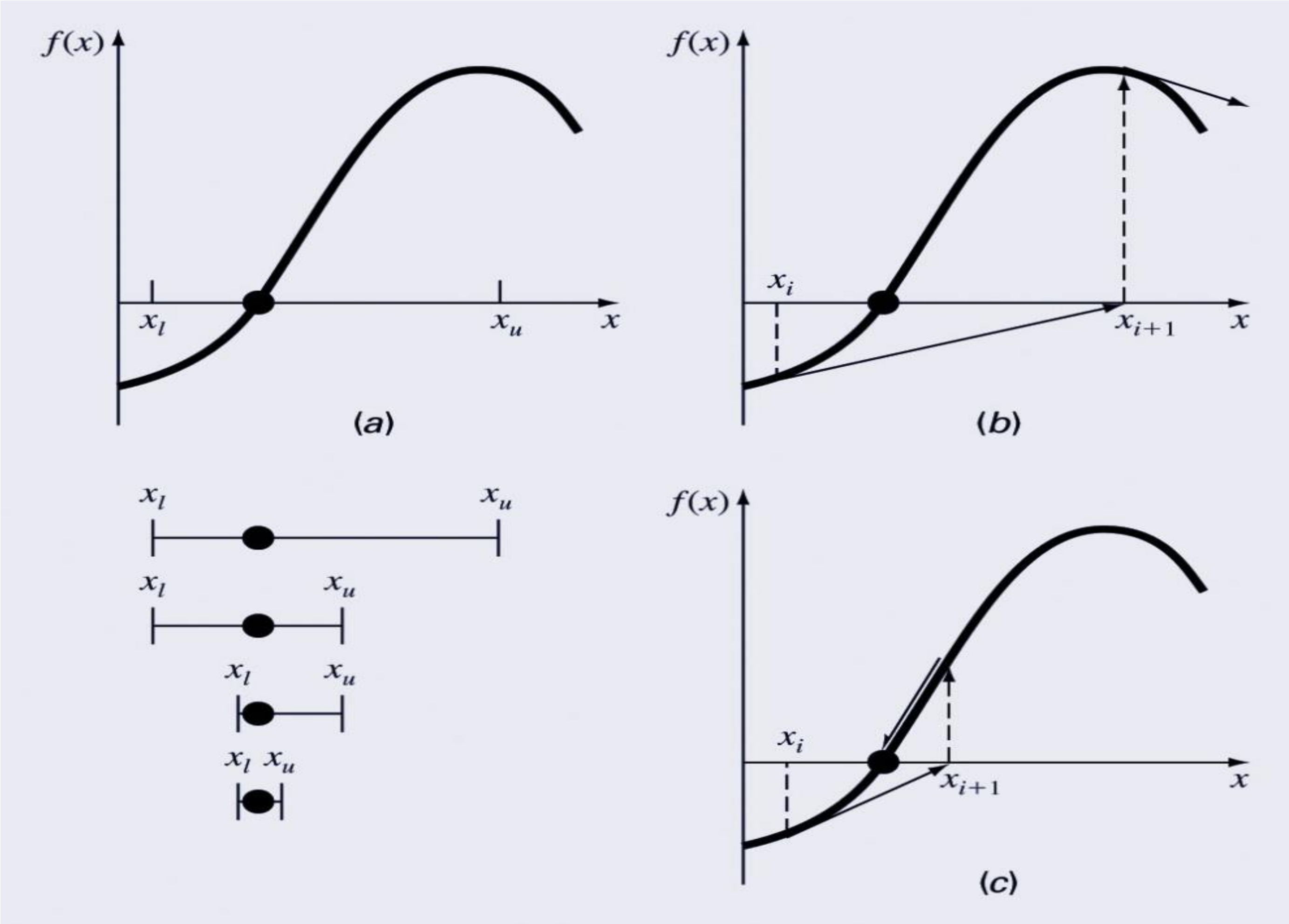
$$\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| 100\%$$

    is smaller than a pre-specified value.
  - When used
    - The only alternative for non-linear systems of equations;
    - Often useful even for linear problems involving a large number of variables where direct methods would be prohibitively expensive or impossible.
- **Convergence of a numerical methods →** if successive approximations lead to increasingly smaller relative error. Opposite to divergent.

Iterative methods for finding the roots

- *Bracketing methods*
- *Open methods*:
  - require only a single starting value or two starting values that do not necessarily bracket a root.
  - may diverge as the computation progresses, but when they do converge, they usually do so much faster than bracketing methods.

Bracketing vs Open; Convergence vs Divergence



a) Bracketing method → start with an interval.
Open method → start with a single initial guess

- b) Diverging open method
- c) Converging open method - note speed!

Simple Fixed-Point Iteration

- Rearrange the function $f(x)=0$ so that x is on the left-hand side of the equation:

$$x=g(x)$$

- Use the new function g to predict a new value of x. The recursion equation:

$$x_{i+1}=g(x_i)$$

- The approximate error is:

$$\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| 100\%$$

- Graphically, the root is at the intersection of two curves:

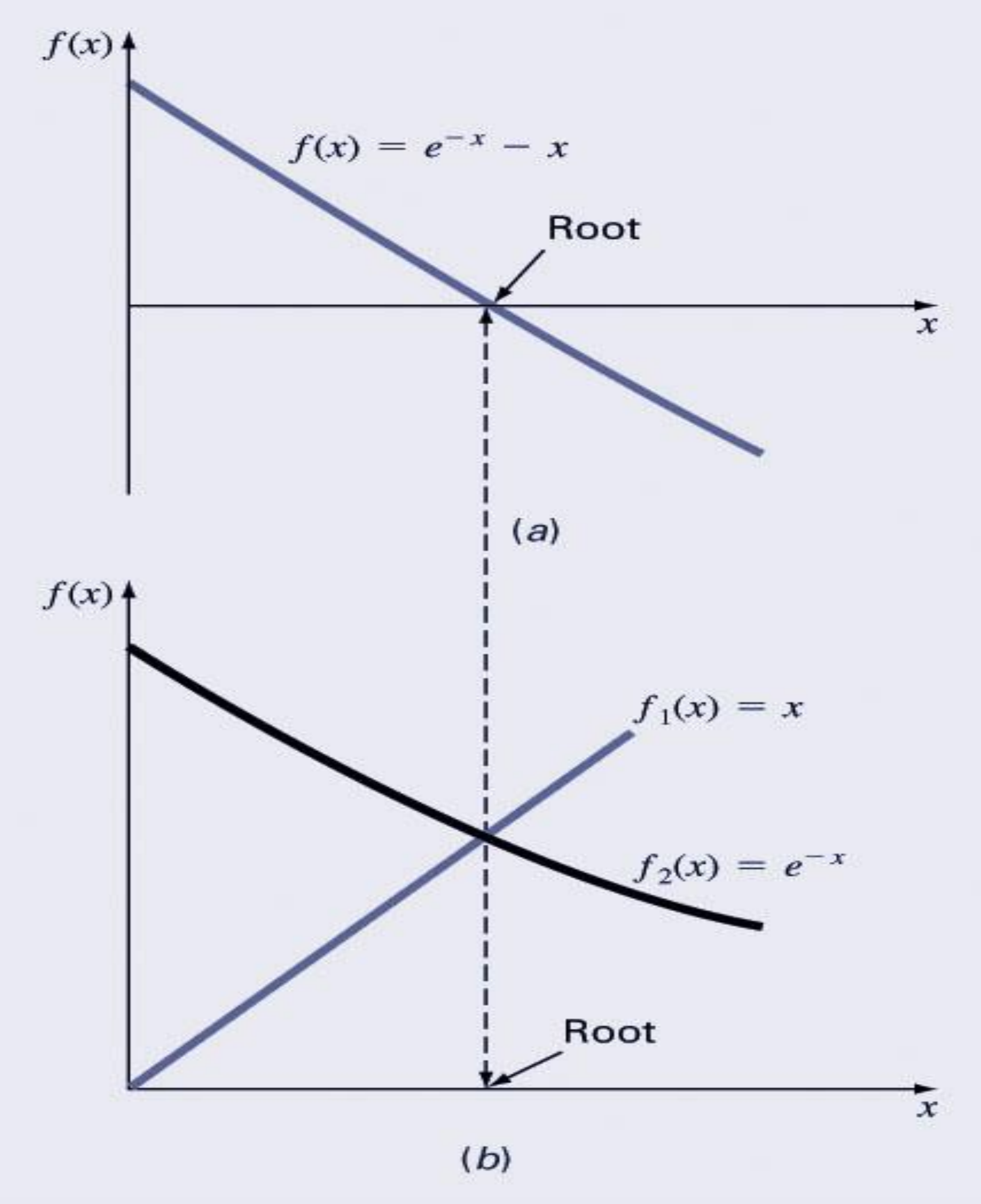
$$y_1(x) = g(x)$$
$$y_2(x) = x.$$

Example

- Solve $f(x)=e^{-x}-x$
- Re-write as: $x=g(x)$ ➔ $x=e^{-x}$
- Start with an initial guess (here, 0)

i	x_i	$\lvert\varepsilon_a\rvert$ %	$\lvert\varepsilon_t\rvert$ %	$\lvert\varepsilon_t\rvert_i/\lvert\varepsilon_t\rvert_{i-1}$
0	0.0000		100.000	
1	1.0000	100.000	76.322	0.763
2	0.3679	171.828	35.135	0.460
3	0.6922	46.854	22.050	0.628
4	0.5005	38.309	11.755	0.533

- Continue until some tolerance is reached



More on Convergence

- Graphically→ the solution is at the intersection of the two curves. We identify the point on y_2 corresponding to the initial guess and the next guess corresponds to the value of the argument x where $y_1(x) = y_2(x)$.

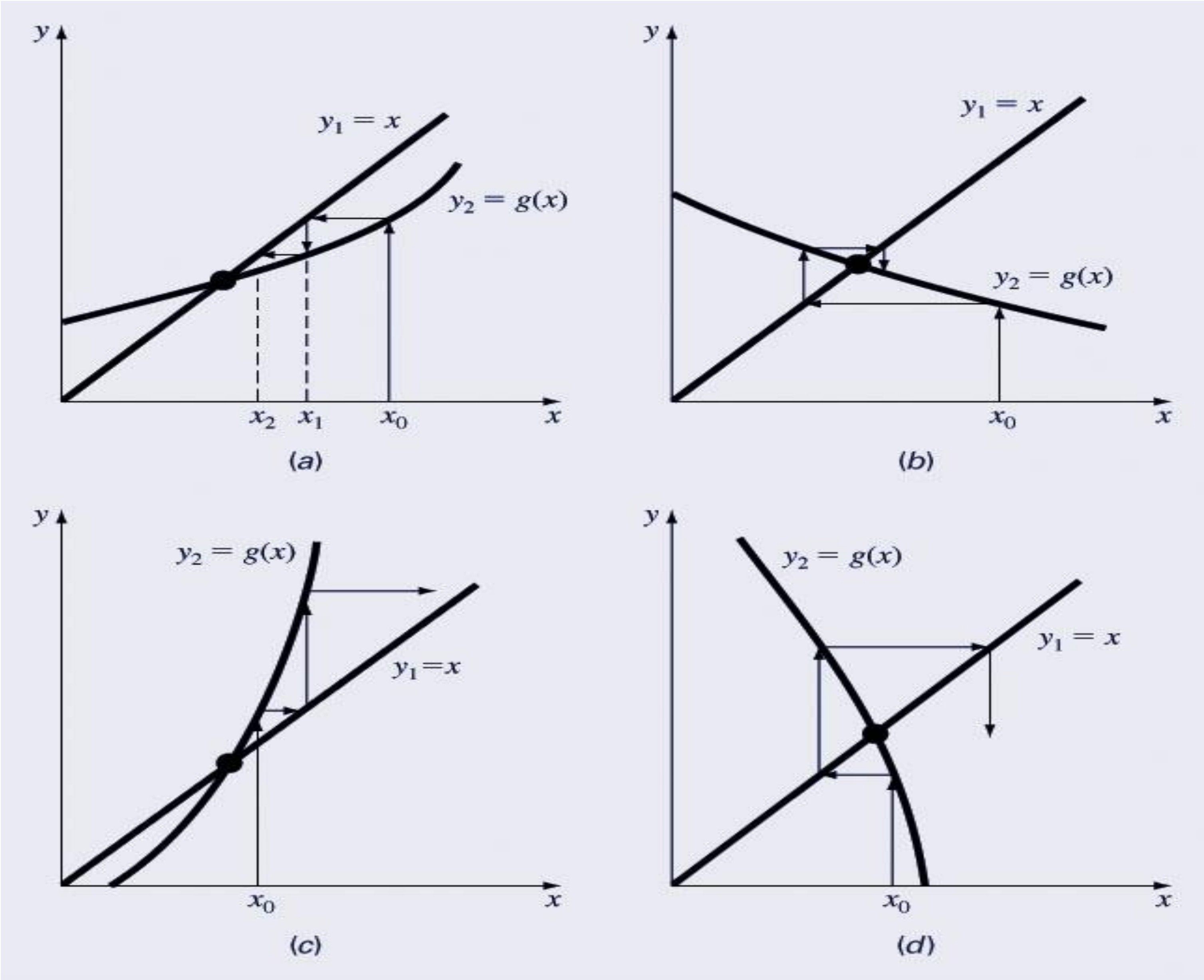
- Convergence of the simple fixed-point iteration method requires that the derivative of g(x) near the root has a magnitude less than 1.

  a) Convergent, $0 \le g' < 1$

  b) Convergent, $-1 < g' \le 0$

  c) Divergent, $g' > 1$

  d) Divergent, $g' < -1$



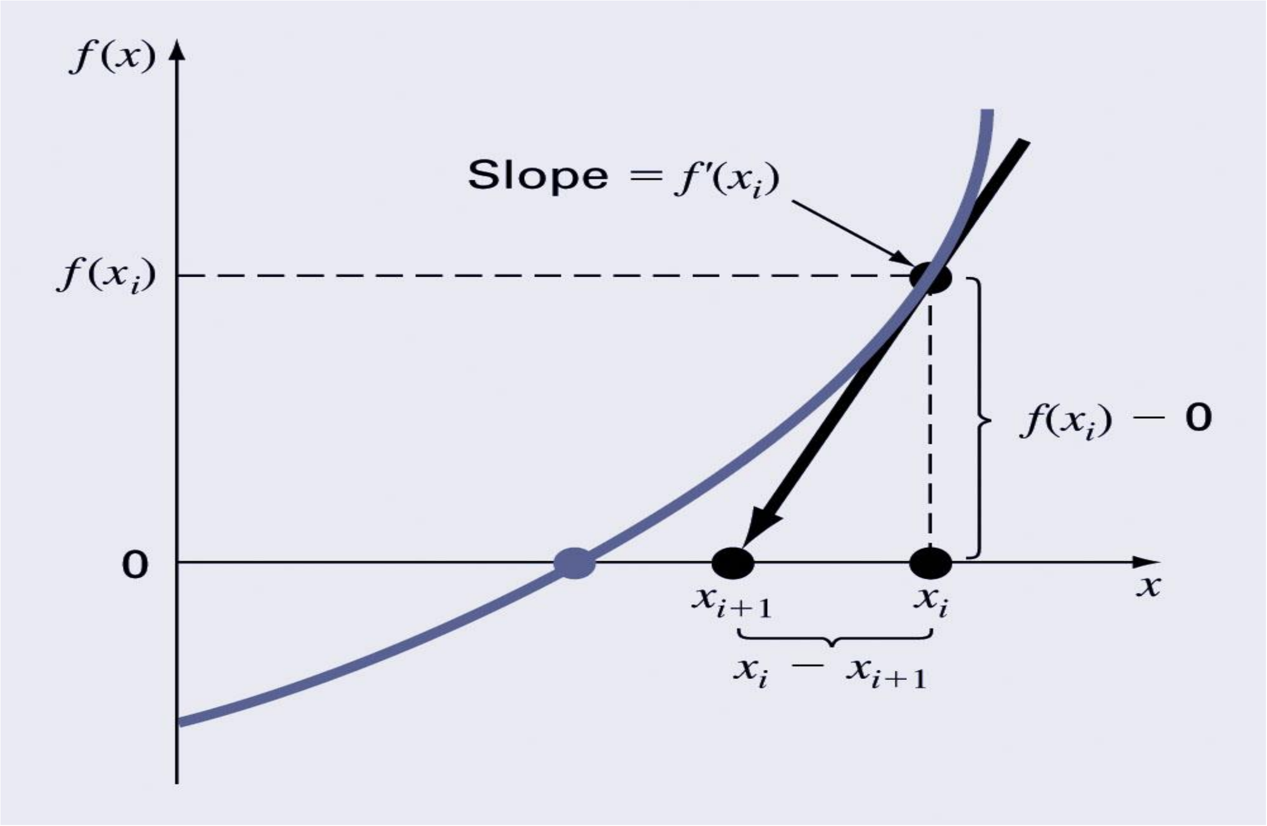
Newton-Raphson Method

- Express x_{i+1} function of x_i and the values of the function and its derivative at x_i.

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- Graphically → draw the tangent line to the *f*(*x*) curve at some guess *x*, then follow the tangent line to where it crosses the *x*-axis.



Numerical Solution of Equations

- This presentation covers the numerical solution of equations for all A Level pure mathematics syllabuses.

- The methods described are

  Change of Sign - Decimal Search
  Fixed Point Iteration - rearranging a formula
  Newton-Raphson Iteration

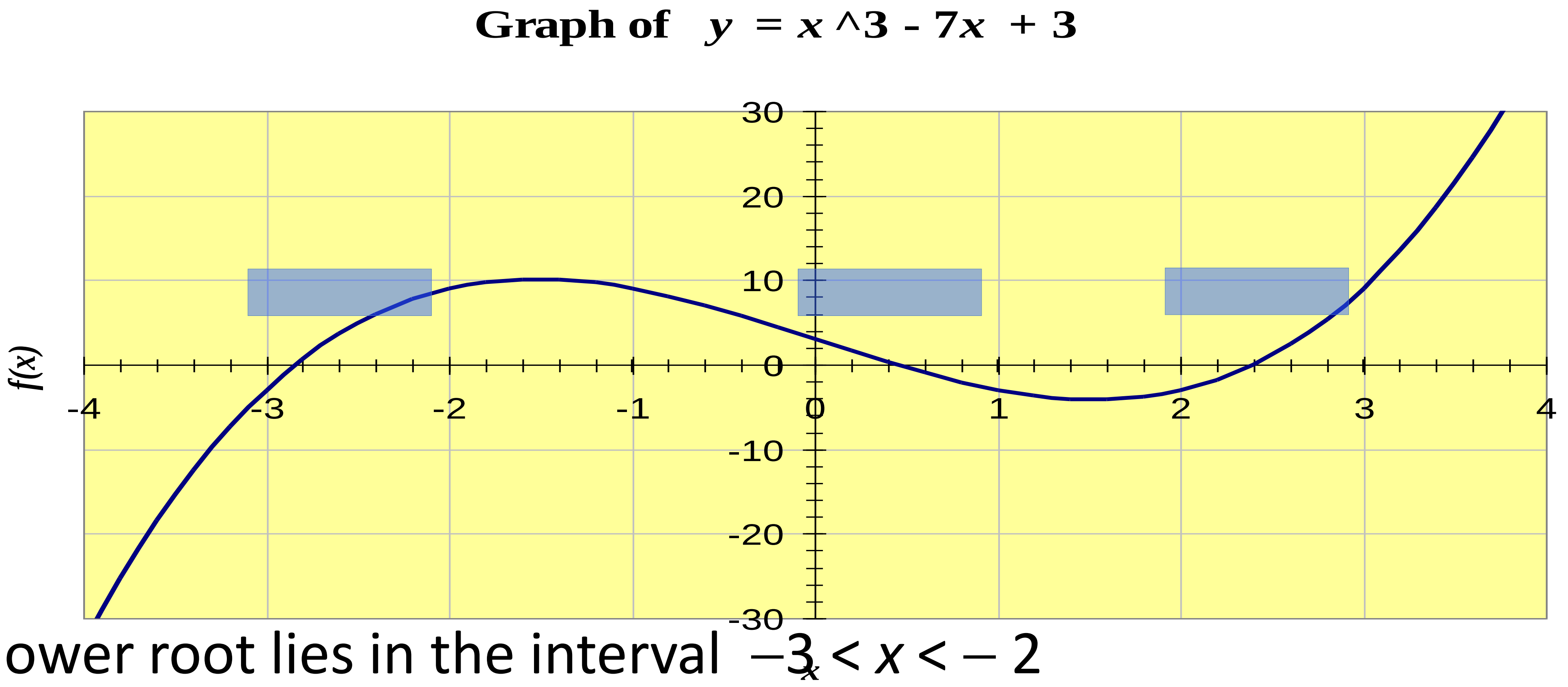
- Most slides proceed with automatic timing.
- Use the mouse to click on a button after each slide.

Why a numerical method ?

An equation f(x) = 0, where f(x) = $x^3 - 7x + 3$, has 3 real roots, but there is no simple analytical method of finding them.

The table and graph suggest first approximations to the roots of the equation $x^3 - 7x + 3 = 0$.

x	f(x)
-4	-33
-3	-3
-2	9
-1	9
0	3
1	-3
2	-3
3	9
4	39



The lower root lies in the interval $-3 < x < -2$

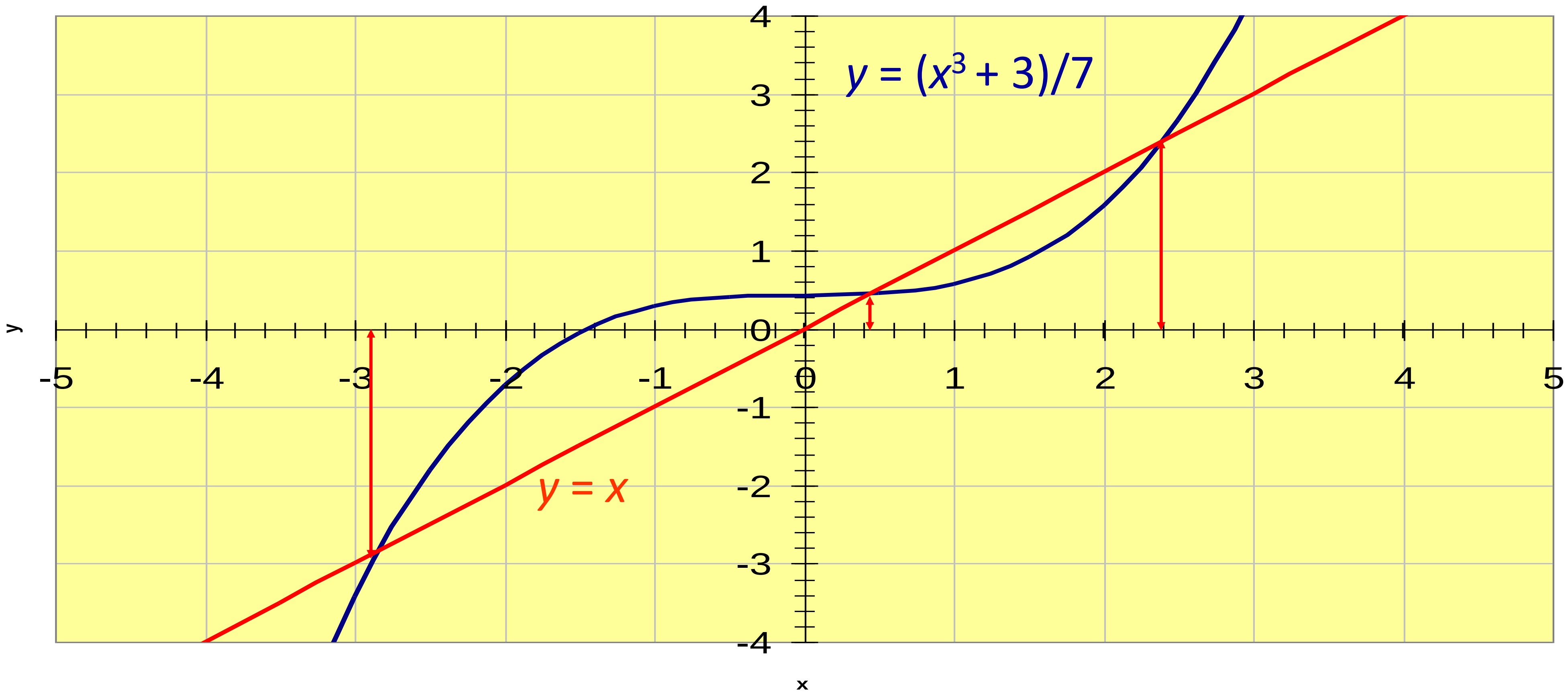
The middle root lies in the interval $0 < x < 1$

The upper root lies in the interval $2 < x < 3$

Fixed Point Iteration

The equation f(x) = 0, where f(x) = $x^3 - 7x + 3$, may be re-arranged to give $x = (x^3 + 3)/7$.

Intersection of the graphs of $y = x$ and $y = (x^3 + 3)/7$ represent roots of the original equation $x^3 - 7x + 3 = 0$.



Fixed Point Iteration

The rearrangement $x = (x^3 + 3)/7$ leads to the iteration

$$x_{n+1} = \frac{x_n^{\,3} + 3}{7}, \quad n = 0, 1, 2, 3, \ldots$$

To find the middle root α, let initial approximation $x_0 = 2$.

$$x_1 = \frac{x_0^{\,3} + 3}{7} = \frac{2^3 + 3}{7} = 1.57143$$

$$x_2 = \frac{x_1^{\,3} + 3}{7} = \frac{1.57143^3 + 3}{7} = 0.98292$$

$$x_3 = \frac{x_2^{\,3} + 3}{7} = \frac{0.98292^3 + 3}{7} = 0.56423$$

$$x_4 = \frac{x_3^{\,3} + 3}{7} = \frac{0.56423^3 + 3}{7} = 0.45423$$

etc.

The iteration slowly converges to give α = **0.441** (to 3 s.f.)

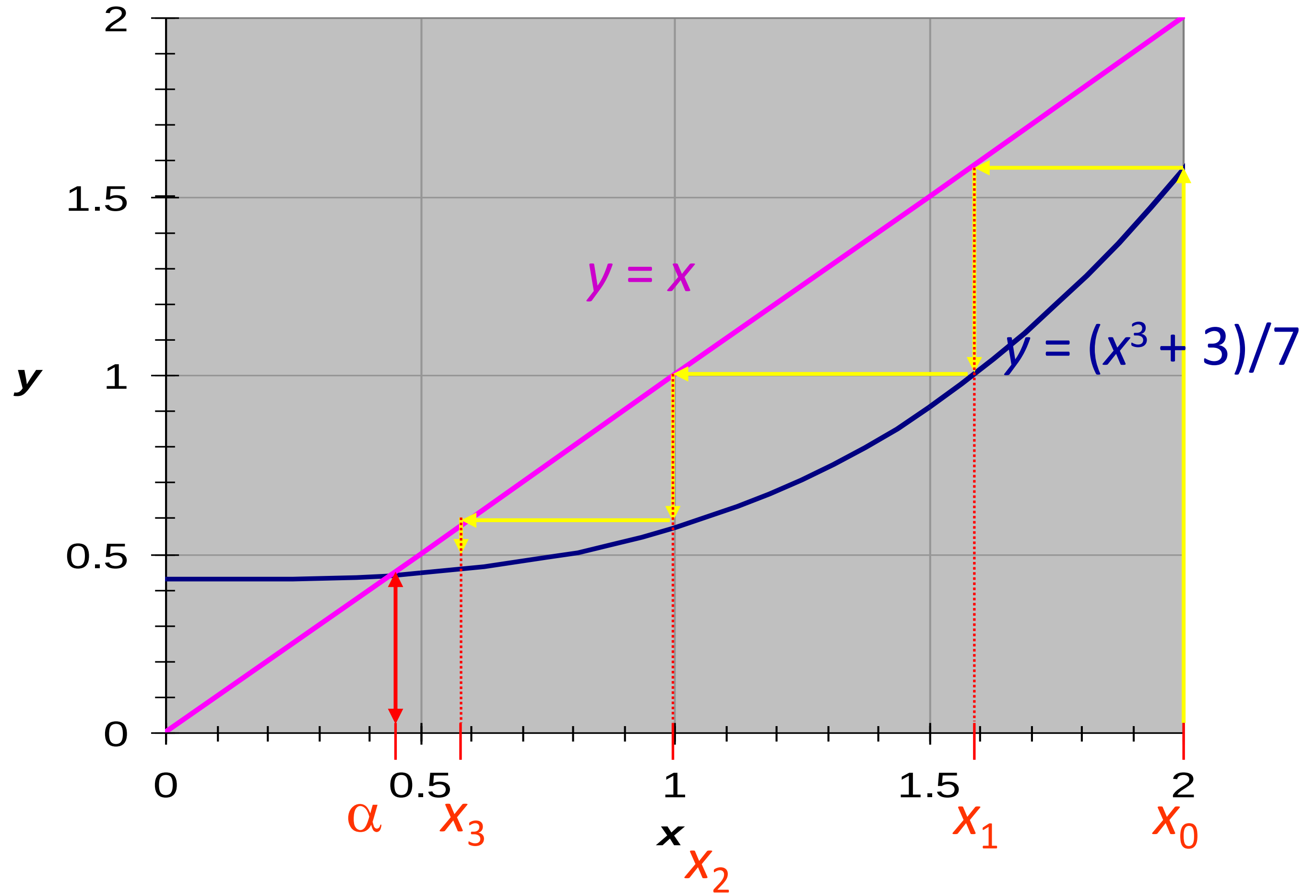
Fixed Point Iteration

The rearrangement $x = (x^3 + 3)/7$ leads to the iteration

$$x_{n+1} = \frac{x_n^{\ 3} + 3}{7}, \quad n = 0, 1, 2, 3, \ldots$$

For $x_0 = 2$ the iteration will converge on the middle root α, since g'(α) < 1.

n	x_n
0	2
1	1.57143
2	0.98292
3	0.56423
4	0.45423
5	0.44196
6	0.4409
7	0.44082
8	0.44081



α = **0.441** (to 3 s.f.)

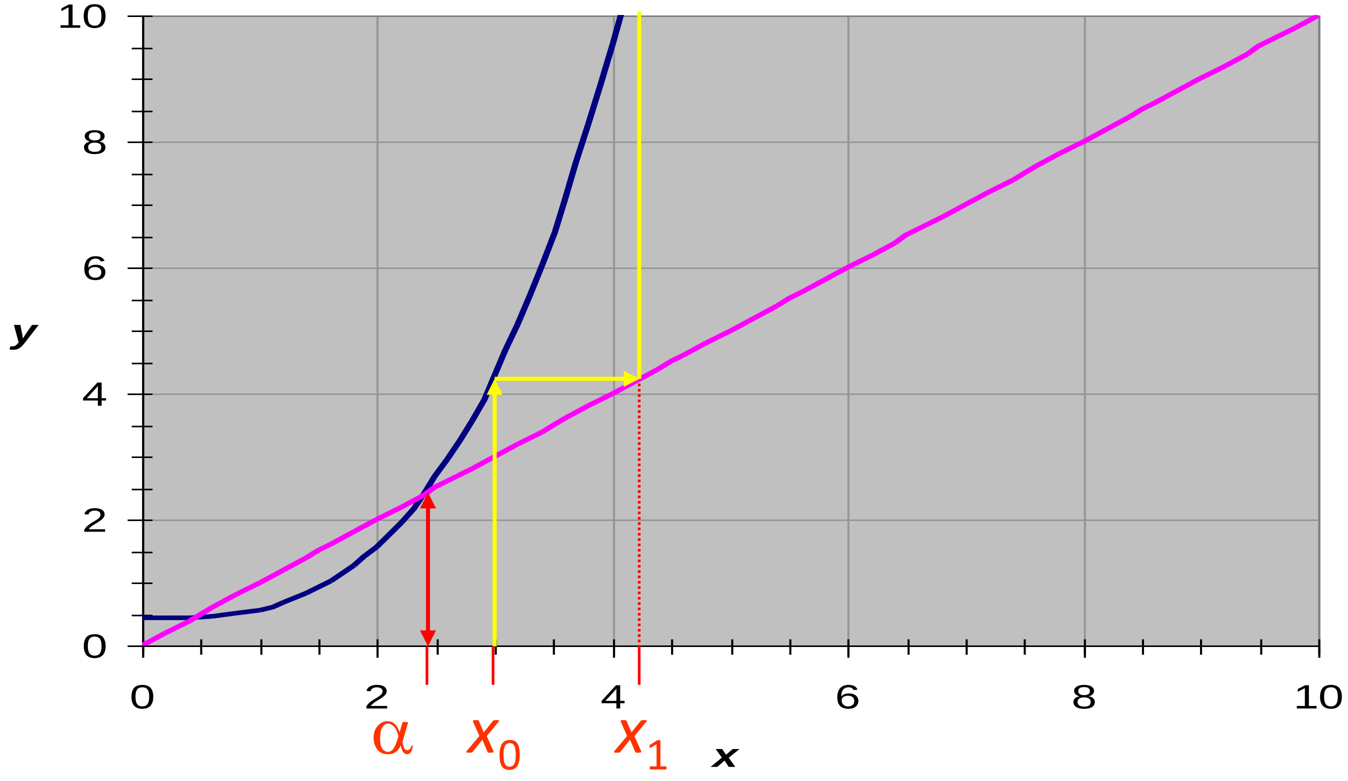
Fixed Point Iteration - breakdown

The rearrangement $x = (x^3 + 3)/7$ leads to the iteration

$$x_{n+1} = \frac{x_n^{\ 3} + 3}{7}, \quad n = 0, 1, 2, 3, \ldots$$

For $x_0 = 3$ the iteration will diverge from the upper root α.

n	x_n
0	3
1	4.28571
2	11.6739
3	227.702
4	1686559
5	6.9E+17



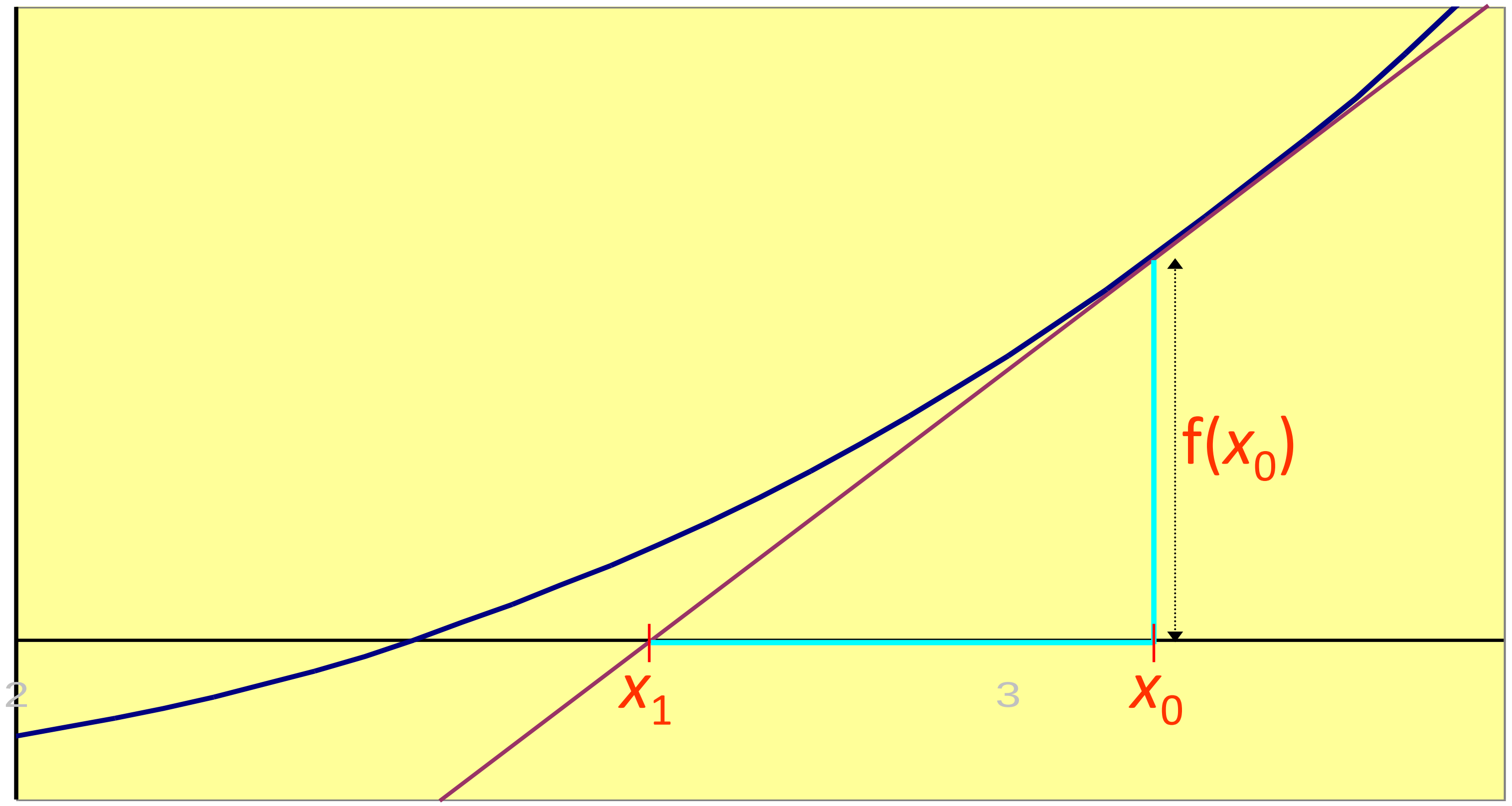
The iteration diverges because $g'(\alpha) > 1$.

NEWTON-RAPHSON ITERATION

The Newton Raphson method is based on the iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots$$

with initial approximation x_0.



Gradient of tangent

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Newton-Raphson Iteration

To solve the equation f(x) = 0, where f(x) = $x^3 - 7x + 3$, use the iteration :

$$x_{n+1} = x_n - \frac{x_n^{\ 3} - 7x_n + 3}{3x_n^{\ 2} - 7}, \quad n = 0, 1, 2, 3, \ldots$$

To find the upper root α, let initial approximation x_0 = 3.

$$x_1 = x_0 - \frac{x_0^{\ 3} - 7x_0 + 3}{3x_0^{\ 2} - 7} = 3 - \frac{3^3 - 7\times 3 + 3}{3\times 3^2 - 7} = 2.55$$

$$x_2 = x_1 - \frac{x_1^{\ 3} - 7x_1 + 3}{3x_1^{\ 2} - 7} = 2.55 - \frac{2.55^3 - 7\times 2.55 + 3}{3\times 2.55^2 - 7} = 2.411573$$

$$x_3 = x_2 - \frac{x_2^{\ 3} - 7x_2 + 3}{3x_2^{\ 2} - 7} = 2.41.. - \frac{2.41..^3 - 7\times 2.41.. + 3}{3\times 2.41..^2 - 7} = 2.397795$$

etc.

The iteration quickly converges, giving α **= 2.40** (to 3.s.f.)

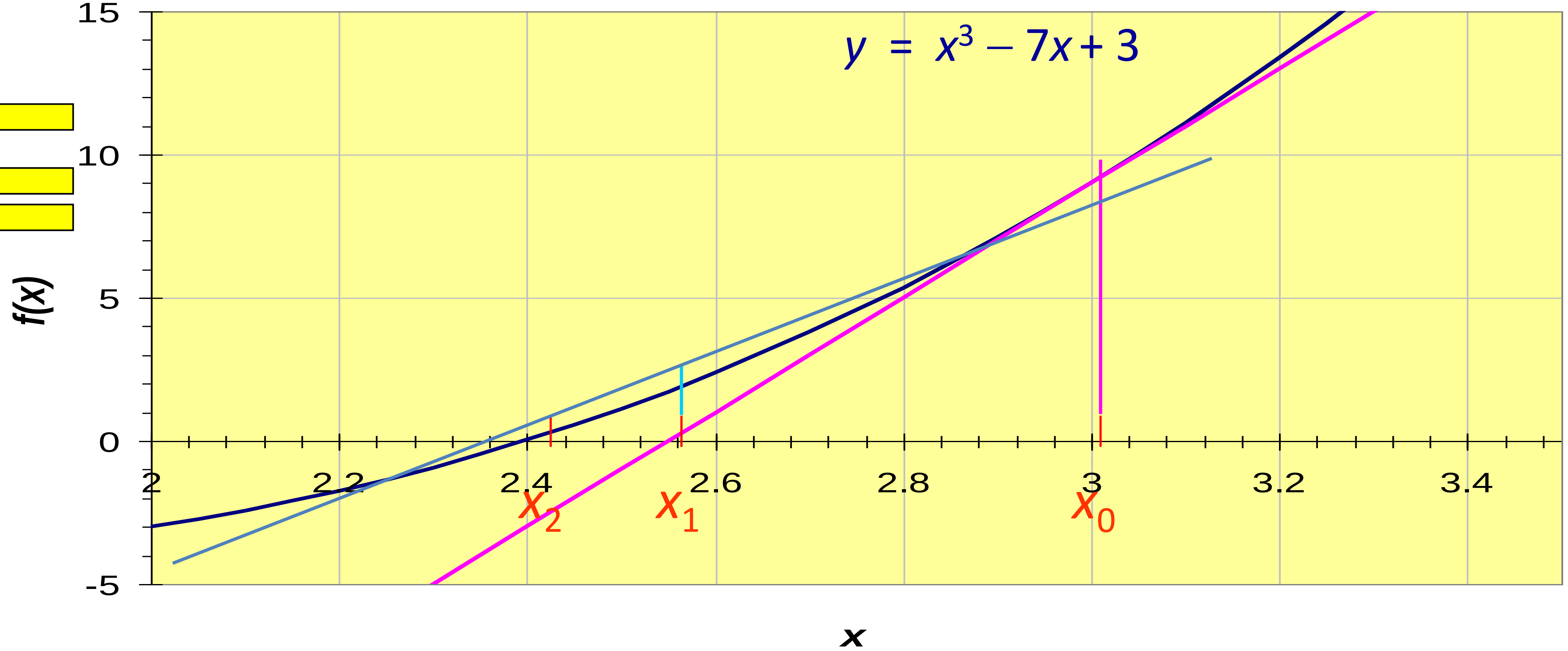
Newton-Raphson Iteration

To solve the equation f(x) = 0, where f(x) = $x^3 - 7x + 3$, use the iteration :

$$x_{n+1} = x_n - \frac{x_n^{\ 3} - 7x_n + 3}{3x_n^{\ 2} - 7}, \quad n = 0, 1, 2, \ldots$$

To find the upper root α, let initial approximation x_0 = 3.

n	x_n
0	3
1	2.55
2	2.411573
3	2.397795
4	2.397662
5	2.397662



The iteration quickly converges, giving **α = 2.40** (to 3.s.f.)

Newton-Raphson Iteration

To solve the equation f(x) = 0, where f(x) = $x^3 - 7x + 3$, use the iteration :

$$x_{n+1} = x_n - \frac{x_n^{\ 3} - 7x_n + 3}{3x_n^{\ 2} - 7}, \quad n = 0, 1, 2, \ldots$$

Choice of intial approximation x_0 will determine which root is found.

n	x_n	x_n	x_n
0	-2	1	3
1	-3.8	0.25	2.55
2	-3.10419	0.43578	2.411573
3	-2.86763	0.440803	2.397795
4	-2.83888	0.440808	2.397662
5	-2.83847	0.440808	2.397662

Initial approximations $x_0 = -2$, $x_0 = 1$ and $x_0 = 3$.

Iterations converge to –2.84, 0.441 and 2.40 respectively (to 3 s.f.)

Newton-Raphson Iteration - breakdown

To solve the equation f(x) = 0, where f(x) = 1/x + 3, use the iteration :

$$x_{n+1} = x_n - \frac{1/x_n + 3}{-1/x_n{}^2}, \quad n = 0, 1, 2, 3, \ldots$$

To find the only root α, let initial approximation $x_0 = -1$.

$$x_1 = x_0 - \frac{1/x_0 + 3}{-1/x_0{}^2} = -1 - \frac{1/(-1) + 3}{-1/(-1)^2} = 1$$

$$x_2 = x_1 - \frac{1/x_1 + 3}{-1/x_1{}^2} = 1 - \frac{1/1 + 3}{-1/1^2} = 5$$

$$x_3 = x_2 - \frac{1/x_2 + 3}{-1/x_2{}^2} = 5 - \frac{1/5 + 3}{-1/5^2} = 85$$

etc. The iteration quickly diverges, failing to give the root α.

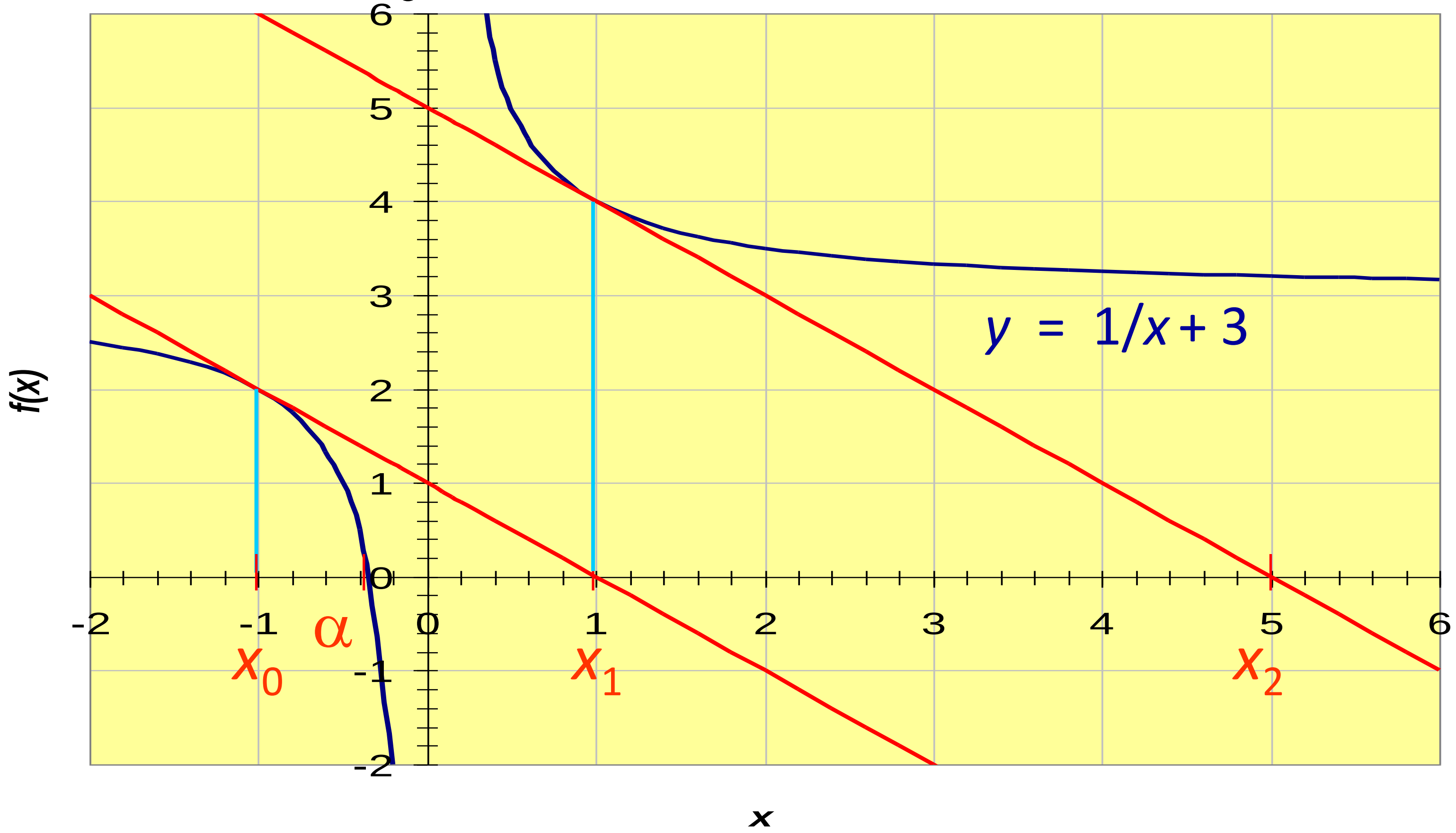
Newton-Raphson Iteration - breakdown

To solve the equation f(x) = 0, where f(x) = 1/x + 3, use the iteration :

$$x_{n+1} = x_n - \frac{1/x_n + 3}{-1/x_n^{\,2}}, \quad n = 0, 1, 2, 3, \ldots$$

To find the only root α, let initial approximation x_0 = −1.

n	x_n
0	-1
1	1
2	5
3	85
4	21845
5	1431655765



The iteration quickly diverges, failing to give the root α = -1/3.